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Higher dimensional geometries related to fuzzy odd-dimensional spheres

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Abstract: We study SO(m) covariant matrix realizations of $\sum_{i=1}^{m} X_i^2 = 1$ for even m as candidate fuzzy odd spheres following [11]. As for the fuzzy four sphere, these matrix algebras contain more degrees of freedom than the sphere itself and the full set of variables has a geometrical description in terms of a higher dimensional coset. The fuzzy S^{2k-1} is related to a higher dimensional coset $\mathrm{SO}(2k)/\mathrm{U}(1) \times \mathrm{U}(k-1)$. These cosets are bundles where base and fibre are hermitean symmetric spaces. The detailed form of the generators and relations for the matrix algebras related to the fuzzy three-spheres suggests matrix actions which admit the fuzzy spheres as solutions. These matrix actions are compared with the BFSS, IKKT and BMN matrix models as well as some others. The geometry and combinatorics of fuzzy odd spheres lead to some remarks on the transverse five-brane problem of matrix theories and the exotic scaling of the entropy of 5-branes with the brane number.

Keywords: Field Theories in Higher Dimensions, D-branes, M(atrix) Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Fuzzy spheres provide many interesting solutions to matrix Brane actions [1]–[6] They have also been suggested to play a role in the context of a spacetime explanation of the stringy exclusion principle [7]–[10].

A class of odd-dimensional fuzzy spheres was defined in [11] and studied in more detail in [12]. The detailed $\mathrm{SO}(m)$ decomposition of the matrix algebras related to the fuzzy S^m was given in [12]. For $m > 2$ the matrix algebras contain more representations than is necessary to describe functions on the sphere, and a projection is needed to get the

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desired traceless symmetric representations. The geometry of general even-dimensional fuzzy spheres was studied in [13] and it was found that the matrix algebras related to the fuzzy sphere of $2k$ dimensions approaches, in a limit of large matrices, the algebra of functions on a higher dimensional space $\mathrm{SO}(2k+1)/\mathrm{U}(k)$, the coset of $\mathrm{SO}(2k+1)$ by the right action of $\mathrm{U}(k)$. This lead to the statement, for $k \geq 2$, that fluctuations around the fuzzy sphere solution can be described equivalently in two ways. On the one hand there is an abelian theory on the higher dimensional coset. On the other hand there is a non-abelian theory on the sphere. At finite n this non-abelian theory on the fuzzy sphere has to be formulated on a commutative but non-associative algebra [12]. The structure constants of the associative algebra for the higher dimensional coset have been described more explicitly in [14]. The abelian theory on the higher dimensional coset, in the case $k = 2$ has been further studied in [15]. A connection between the fuzzy four-sphere and the 4D quantum Hall effect [16] was found in [17] and higher even fuzzy spheres were used to explore 6D and 8D generalizations of the quantum Hall effect. The extra degrees of freedom of the fuzzy four-sphere have been discussed in the context of uncertainty relations in [18].

In this paper we explore the extent to which the picture of physics on even-dimensional fuzzy spheres developed in [13] can be extended to fuzzy odd spheres. We find that, as in the even case, there is an underlying higher dimensional geometry related to the matrix algebra. For the case of the sphere S^{2k-1} with symmetry $\mathrm{SO}(2k)$, the relevant geometry is $\mathrm{SO}(2k)/\mathrm{U}(k-1) \times \mathrm{U}(1)$, the coset of $\mathrm{SO}(2k)$ by the right action of $\mathrm{U}(k-1) \times \mathrm{U}(1)$. We will discuss in detail the cases $k = 2$ and $k = 3$ in this paper but it appears that arguments leading to these cosets should generalize easily, especially the arguments in sections 3.3 and 8.1.

The precise role of the $\mathrm{SO}(2k)/\mathrm{U}(k-1) \times \mathrm{U}(1)$ is, however, different from the role played by the $\mathrm{SO}(2k+1)/\mathrm{U}(k)$ in the case of even-dimensional spheres. To explain this, we need to recall some details about the construction of even and odd fuzzy spheres. In all cases one starts with matrices X_i, where i transforms in the vector of $\mathrm{SO}(m)$, and

$$\sum_{i=1}^{m} X_i^2 = 1\,. \tag{1.1}$$

The X_i are in $\mathrm{End}(\mathcal{R}_n)$, i.e. matrices which are transformations of the vector space $\mathcal{R}_n$, which is a representation of $\mathrm{SO}(m)$. The X_i in (1.1) are related by a rescaling to the X_i used in the bulk of this paper. $\mathcal{R}_n$ is a vector space whose dimension N depends on an integer n. The precise dependence of N on n can be found in [12]. The crucial difference between even and odd spheres is that, in the case of even spheres $\mathcal{R}_n$ is an irreducible representation of $\mathrm{SO}(2k+1)$, whereas in the case of odd spheres $\mathcal{R}_n$ is a direct sum of irreducible representations $\mathcal{R}_n^+$ and $\mathcal{R}_n^-$ of $\mathrm{SO}(2k)$, that is, $\mathcal{R}_n = \mathcal{R}_n^+ \oplus \mathcal{R}_n^-$. The matrices X_i are maps from $\mathcal{R}_n^+$ to $\mathcal{R}_n^-$ and vice versa from $\mathcal{R}_n^-$ to $\mathcal{R}_n^+$.

For even fuzzy spheres, $\mathrm{End}(\mathcal{R}_n)$ becomes commutative in the large-n limit and approaches the algebra of functions on a classical space $\mathrm{SO}(2k+1)/\mathrm{U}(k)$. In the odd sphere case, both $\mathrm{End}(\mathcal{R}_n^+)$ and $\mathrm{End}(\mathcal{R}_n^-)$ become commutative in the large-n limit and approach the algebra of functions on $\mathrm{SO}(2k)/\mathrm{U}(k-1) \times \mathrm{U}(1)$. However the full matrix algebra $\mathrm{End}(\mathcal{R}_n)$ remains non commutative. In the case $k = 2$, the relevant higher dimensional

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space $SO(4)U(1) \times U(1)$ has a simpler description as $S^2 \times S^2$ when we use the isomorphism $SO(4) \equiv SU(2) \times SU(2)$. Section 3 gives two ways to prove that $\mathrm{End}(\mathcal{R}_n^{\pm})$ approaches the algebra of functions on $S^2 \times S^2$ in the large-N limit. One uses the techniques of Kaluza-Klein reduction [19] or equivalently harmonic analysis on homogeneous spaces, as described for example in [20]. Another uses the analysis of the stabilizer group of a particular solution to the algebraic relations satisfied by the generators of the matrix algebra. Section 8 proves, by similar methods, that the higher dimensional geometry related to the fuzzy five-sphere is $SO(6)U(2) \times U(1)$.

In sections 4–7 we explore the fuzzy three-sphere in more detail. Section 4 gives the relations between the generators of $\mathrm{End}(\mathcal{R}_n)$. Section 5 gives the large-n limit of these relations, showing, in particular, that while $\mathrm{End}(\mathcal{R}_n^{\pm})$ become commutative, $\mathrm{End}(\mathcal{R}_n)$ remains non commutative. Section 6 uses the relations derived in sections 4 and 5 to construct some matrix actions which are solved by the fuzzy three-sphere matrices. Section 7 discusses the physics of fluctuations around these solutions.

In section 9.1 we discuss the geometry of these cosets $SO(2k)U(k-1) \times U(1)$ further. In particular with describe some bundle structures they admit. For the even sphere case, the bundle structure of $SO(2k+1)/\,U(k)$ was used to map abelian field theory on the fuzzy coset to non-abelian field theory on the fuzzy sphere base [13]. In the odd sphere case, the relevant higher dimensional geometry is not in general a bundle over the sphere, so the same strategy as [13] cannot be used to obtain a field theory on the sphere. However some bundle structures on the coset do exist and we describe them. An interesting feature is that base and fibre are both hermitean symmetric spaces, which come up in $N = 2$ sigma models for example [21]. In section 9.3 we give a qualitative comparison of some basic counting of degrees of freedom which explains why it appears dificult to obtain an ordinary non-abelian theory on the fuzzy odd sphere. We observe some intriguing similarities to discussions of the entropy of M5-branes. Section 9 also describes some avenues for future research.

2. Review of fuzzy odd spheres

To get a fuzzy three sphere [11] we look for matrices which satisfy the equation $\sum_i X_i^2 = c$, where c is a constant. The matrices are constructed by starting with $V = V_+ \oplus V_-$. The vector space V_+ is the two-dimensional spinor representation of $SO(4)$ of positive chirality. The space V_- is the two-dimensional representation of negative chirality. The projector P_+ acting on V projects onto V_+, and P_- projects onto V_-. In terms of the $SO(4) = SU(2) \times SU(2)$ isomorphism, these have spins $(2j_L, 2j_R) = (1, 0)$ and $(2j_L, 2j_R) = (0, 1)$, respectively. For every odd integer n, the space $\mathrm{Sym}(V^{\otimes n})$ (which is the symmetrized n-fold tensor product space) contains a subspace where there are $(n+1)/2$ factors of positive chirality and $(n-1)/2$ factors of negative chirality. This subspace is an irreducible representation of $SO(4)$ labelled by $(2j_L, 2j_R) = ((n+1)/2, (n-1)/2)$, which we will call $\mathcal{R}_n^+$. The projection operator in $\mathrm{End}(\mathrm{Sym}(V^{\otimes n}))$ which projects onto this irrep is called $\mathcal{P}_{\mathcal{R}_n^+}$. In [11] it was called $\mathcal{P}_{\mathcal{R}^+}$, but here we are using notation which makes the n dependence explicit. $\mathrm{Sym}(V^{\otimes n})$ also contains a subspace where there are $(n+1)/2$ factors of negative chirality

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and $(n-1)/2$ factors of positive chirality. This subspace is an irreducible representation of SO(4) labelled by $(2j_L, 2j_R) = ((n-1)/2, (n+1)/2)$, which we will call $\mathcal{R}_n^-$. The projector for this subspace is called $\mathcal{P}_{\mathcal{R}_n^-}$. The space $\mathcal{R}_n$ is defined to be a direct sum

$$\mathcal{R}_n = \mathcal{R}_n^+ \oplus \mathcal{R}_n^- . \tag{2.1}$$

The projector for this space is a sum

$$\mathcal{P}_{\mathcal{R}_n} = \mathcal{P}_{\mathcal{R}_n^+} + \mathcal{P}_{\mathcal{R}_n^-} . \tag{2.2}$$

The matrices X_i are linear transformations of $\mathcal{R}_n$, i.e. they are in End$(\mathcal{R}_n)$.

$$X_i = \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} . \tag{2.3}$$

More precisely, X_i map $\mathcal{R}_n^+$ to $\mathcal{R}_n^-$ and $\mathcal{R}_n^-$ to $\mathcal{R}_n^+$. This can be expressed by saying it is a sum of matrices in Hom$(\mathcal{R}_n^+, \mathcal{R}_n^-)$ and in Hom$(\mathcal{R}_n^+, \mathcal{R}_n^-)$

$$X_i = \mathcal{P}_{\mathcal{R}_n^+} X_i \mathcal{P}_{\mathcal{R}_n^-} + \mathcal{P}_{\mathcal{R}_n^-} X_i \mathcal{P}_{\mathcal{R}_n^+} . \tag{2.4}$$

In other words, if we arrange the vectors of $\mathcal{R}^+$ along the upper rows of a column vector and those of $\mathcal{R}^-$ along the lower rows, the matrices X_i are non zero in the off-diagonal blocks.

Let us prove that $\sum_i X_i^2$ commutes with generators of SO(4).

$$\begin{aligned}
&\left[\sum_i X_i^2, \mathcal{P}_{\mathcal{R}_n} \sum_t \rho_t(\Gamma_k \Gamma_l) \mathcal{P}_{\mathcal{R}_n}\right] = \\
&\quad = \sum_i \left[X_i, \mathcal{P}_{\mathcal{R}_n} \sum_t \rho_t(\Gamma_k \Gamma_l) \mathcal{P}_{\mathcal{R}_n}\right] X_i + X_i \left[X_i, \mathcal{P}_{\mathcal{R}_n} \sum_t \rho_t(\Gamma_k \Gamma_l) \mathcal{P}_{\mathcal{R}_n}\right] \\
&\quad = \sum_i \mathcal{P}_{\mathcal{R}_n} \sum_r \left[\rho_r(\Gamma_i), \sum_t \rho_t(\Gamma_k \Gamma_l)\right] \mathcal{P}_{\mathcal{R}_n} \sum_s \rho_s(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} + \\
&\qquad + \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} \left[\sum_s \rho_s(\Gamma_i), \sum_t \rho_t(\Gamma_k \Gamma_l)\right] \mathcal{P}_{\mathcal{R}_n} \\
&\quad = \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r\left([\Gamma_i, \Gamma_k \Gamma_l]\right) \mathcal{P}_{\mathcal{R}_n} \sum_s \rho_s(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} + \\
&\qquad + \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} \sum_s \rho_s\left([\Gamma_i, \Gamma_k \Gamma_l]\right) \mathcal{P}_{\mathcal{R}_n} \\
&\quad = \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r\left(\Gamma_l \delta_{ik} - \Gamma_k \delta_{il}\right) \mathcal{P}_{\mathcal{R}_n} \sum_s \rho_s(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} + \\
&\qquad + \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} \sum_s (\rho_s(\Gamma_l) \delta_{ik} - \rho_s(\Gamma_k) \delta_{il}) \mathcal{P}_{\mathcal{R}_n} = 0 .
\end{aligned} \tag{2.5}$$

Since X_i^2 commutes, we know it is a constant in each irrep.

$$X_i^2 = a_+ \mathcal{P}_{\mathcal{R}_n^+} + a_- \mathcal{P}_{\mathcal{R}_n^-} . \tag{2.6}$$

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Now consider the action on X_i^2 on R_+. We find

$$X_i^2 \mathcal{P}_{\mathcal{R}_n^+} = \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_i P_-) \mathcal{P}_{\mathcal{R}_n^-} \sum_s \rho_s(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} = a_+ \mathcal{P}_{\mathcal{R}_n^+} \tag{2.7}$$

and likewise:

$$X_i^2 \mathcal{P}_{\mathcal{R}_n^+} = \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} \sum_s \rho_s(\Gamma_i P_-) \mathcal{P}_{\mathcal{R}_n^-} = a_- \mathcal{P}_{\mathcal{R}_n^-} . \tag{2.8}$$

Now consider the operation of exchanging P_+ with P_-, which leaves X_i^2 as defined in (2.3) invariant. Now peform this operation on the eq. (2.7) to find

$$X_i^2 \mathcal{P}_{\mathcal{R}_n^-} = \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} \sum_s \rho_s(\Gamma_i P_-) \mathcal{P}_{\mathcal{R}_n^-} = a_+ \mathcal{P}_{\mathcal{R}_n^-} . \tag{2.9}$$

Comparing with (2.8) we find that $a_+ = a_-$. Thus X_i^2 is proportional to the identity.

The radius is easily calculated as follows:

$$X_i^2 \mathcal{P}_{\mathcal{R}_n^+} = \sum_r \rho_r(\Gamma_i \Gamma_i P_+) + \sum_{r \neq s} \rho_r(\Gamma_i P_-) \rho_s(\Gamma_i P_+) . \tag{2.10}$$

In the second term, the index s can take $(n+1)/2$ vlues for a non-zero answer and the index r can take $(n-1)/2$ values. Also we have $\sum_i (\Gamma_i \otimes \Gamma_i) P_- \otimes P_+ = 2(P_+ \otimes P_-)$. The first term gives $\frac{(n+1)}{2} \times 4$. Adding these two contributions we get, $\frac{(n+1)}{2}(4+(n-1)) = (n+1)(n+3)/2$.

Note that

$$\begin{aligned} &\mathcal{P}_{\mathcal{R}_n} \sum_i \sum_r \rho_r(\Gamma_i) \sum_s \rho_s(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} = \\ &= \sum_i X_i^2 + \mathcal{P}_{\mathcal{R}_n} \sum_i \sum_r \rho_r(\Gamma_i)(\mathcal{P}_{\mathcal{R}_{n++}} + \mathcal{P}_{\mathcal{R}_{n--}}) \sum_s \rho_s(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} , \end{aligned} \tag{2.11}$$

where $\mathcal{P}_{\mathcal{R}_{n++}}$ projects onto $(2j_L, 2j_R) = (\frac{n+3}{2}, \frac{n-3}{2})$, and $\mathcal{P}_{\mathcal{R}_{n--}}$ projects onto $(2j_L, 2j_R) = (n-32, \frac{n+3}{2})$. We should be careful to distinguish the l.h.s. of (2.11) from X_i^2.

The above discussion generalizes easily to fuzzy five spheres [12]. A similar calculation gives

$$\sum_i X_i^2 = \frac{(n+1)}{2}(n+2k-1) . \tag{2.12}$$

3. Fuzzy three-sphere and $S^2 \times S^2$

The matrix Algebras related to fuzzy S^m, for $m > 2$ contain more representations than required to give the algebra of functions on a sphere [12]. In the even sphere case, the higher dimensional manifolds related to the matrix algbras have been identified [13]. We recall some relevant facts here. In the case of the even fuzzy sphere S^{2k} it was just $\mathrm{SO}(2k+1)/\mathrm{U}(k)$. Several different proofs of this result were given. The first proof follows immediately once we know [12] that the matrix algebra contains every representation of $\mathrm{SO}(2k+1)$ with unit multiplicity. We then used the result of [22] that the algebra of

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functions on $\mathrm{SO}(2k+1)/\mathrm{U}(k)$ has precisely this property. In section 3.1 we will follow this line of argument, i.e. we will compare the SO(4) decomposition of the functions on $S^2 \times S^2$ to the representations found in $\mathrm{End}(\mathcal{R}_n^{\pm})$ at large n and find agreement. In section 3.2 we will develop an analogous argument for $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$ or $\mathrm{Hom}(\mathcal{R}_n^-, \mathcal{R}_n^+)$, showing that the space of sections of a bundle on $S^2 \times S^2$ agrees with the SO(4) content of these off-diagonal matrices. A second line of argument to relate fuzzy even spheres to the appropriate coset will be briefly described in section 3.3 and the analog will be developed for the fuzzy three-sphere.

3.1 Spectrum of $\mathrm{End}(\mathcal{R}_n^{\pm})$ and functions on $S^2 \times S^2$

Rotations L_{12} and L_{34} form a $\mathrm{U}(1) \times \mathrm{U}(1)$ subgroup of SO(4). In a self dual representation associated with an SO(4) Young diagram of row lengths (r_1, r_2) the eigenvalues of these generators on the highest weight vector are r_1, r_2. In an anti-self dual representation associated with the same Young diagram the eigenvalues on the highest weight vector are $(r_1, -r_2)$. The combination $L_{12} + L_{34}$ generates a U(1) subgroup of the left SU(2) in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ description of SO(4). The $\mathrm{SU}_L(2)$ weight is, therefore, described by $2J_L = L_{12} + L_{34}$. The combination $L_{12} - L_{34}$ generates a U(1) subgroup of the right SU(2). The $\mathrm{SU}_R(2)$ highest weight is, therefore, described by $2J_R = L_{12} - L_{34}$. For self-dual reps, we have then $2J_L = r_1 + r_2$ and $2J_R = r_1 - r_2$, where r_2 is positive. For antiself-dual reps, $2J_L = r_1 - r_2$ and $2J_R = r_1 + r_2$ for positive r_2.

To get the spectrum of representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ present among the harmonics of $(\mathrm{SU}(2) \times \mathrm{SU}(2))/(\mathrm{U}(1) \times \mathrm{U}(1))$ we consider the representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ induced from the trivial representation of $\mathrm{U}(1) \times \mathrm{U}(1)$. By Frobenius duality, the multiplicity of an irrep. of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in the induced rep. is equal to the multiplicity of the identity rep. in the restriction of that irrep to $\mathrm{U}(1) \times \mathrm{U}(1)$. This allows us to recover the well-known fact that the spectrum of reps is $(2j_L, 2j_R)$ with $2j_L$ and $2j_R$ even.

Reps of highest weight (r_1, r_2) with r_2 positive (i.e. self-dual reps) and $r_1 - r_2$ even, is the same as the set of reps with $2j_L = r_1 + r_2 = (r_1 - r_2) + 2r_2$ even, and $2j_R = r_1 - r_2$ even, with the restriction that $2j_L \geq 2j_R$. Reps of highest weight $(r_1, -r_2)$ with r_2 positive (i.e. anti-self-dual reps) and $r_1 - r_2$ even is the same as the set described by $2j_L$ even and $2j_R$ even with $2j_R \geq 2j_L$. These sets of reps. of SO(4) are the ones which appear in the SO(4) decompositon of $\mathrm{End}(\mathcal{R}_n^+)$ or $\mathrm{End}(\mathcal{R}_n^-)$ at large n [12]. We conclude that large-n SO(4) decomposition of $\mathrm{End}(\mathcal{R}_n^+)$ or $\mathrm{End}(\mathcal{R}_n^+)$ has the same representations as the space of functions on $S^2 \times S^2$. Since the structure constants of the algebra are determined by the Clebsch-Gordan coefficients, once we have established that the matrix algebra has the same repesentations as $Fun(S^2 \times S^2)$ we know that it is the same algebra.

3.2 Spectrum of $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$ and sections on $S^2 \times S^2$

In the large-n limit of $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$ (or $\mathrm{Hom}(\mathcal{R}_n^-, \mathcal{R}_n^+)$) we have self-dual reps associated with Young Diagrams labelled by non-negative row lengths (r_1, r_2) (which have highest weights (r_1, r_2)), and obeying the condition $r_1 - r_2$ odd. We also have antiself-dual reps. associated with Young Diagrams labelled by non-negative row lengths (r_1, r_2) (which have highest weights $(r_1, -r_2)$), and obeying $r_1 - r_2$ odd. These results are found in [12].

This approaches a space of sections on SO(4)/(U(1) $\times$ U(1)). Spaces of sections on this coset are given by induced representations. By the the Frobenius reciprocity theorem, the multiplicity of an irrep of R of G in the rep. induced from H using the irrep. r of H is equal to the multiplicity of r in the restriction of R to H.

The desired space of sections is the one obtained by inducing the irrep of U(1) $\times$ U(1) which corresponds to $(1,0)$. (We could also work with $(-1,0)$ and get the same spectrum as $\mathrm{Hom}(\mathcal{R}_n^-, \mathcal{R}_n^+)$ or $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$). The first U(1) is generated by L_{12} and the second by L_{34}. Writing the product of two U(1) in terms of generators of the two SU(2) factors in the SO(4) $\equiv$ SU(2) $\times$ SU(2) isomorphism, the charges are $(2j_L, 2j_R) = (1,1)$

Consider all reps of SU(2) $\times$ SU(2) which restrict to the $(2j_L, 2j_R) = (1,1)$ of the U(1)$\times$U(1) subgroup. They are all the reps with $(2j_L, 2j_R)$ odd and each contains the $(1,1)$ with unit multiplicity and therefore appears in the space of sections with unit multiplicity. This is exactly the description of the spectrum of SO(4) reps in $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$ from [12] descirbed above.

3.3 $S^2 \times S^2$ by analysis of stabilizer group

Another strategy in to identify the higher domensional coset in [13] was to think about the system of equations obeyed by X_μ and $X_{\mu\nu}$. In particular we needed to find just one solution to the system of equations, and prove that the subgroup of SO(5) which kept that solution invariant was U(2). To find a solution, we took the expectation value of the matrices X_μ and $X_{\mu\nu}$ in one particular state, the n-fold tensor product of the v_0 (see appendix A for notation).

We will do something similar for the fuzzy three-sphere. Identify a set of generators for the complete matrix algebra, and take expectation values in the large-n limit, and analyze the stabilizer group if the expecttaion values.

A complete set of generators is:

$$
\begin{aligned}
X_i^+ &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} \,, \\
X_i^- &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_i P_-) \mathcal{P}_{\mathcal{R}_n^-} \,, \\
X_{ij}^+ &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r \left(\frac{1}{2}[\Gamma_i, \Gamma_j] P_+ \right) \mathcal{P}_{\mathcal{R}_n^+} \,, \\
Y_{ij}^+ &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r \left(\frac{1}{2}[\Gamma_i, \Gamma_j] P_- \right) \mathcal{P}_{\mathcal{R}_n^+} \,, \\
X_{ij}^- &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r \left(\frac{1}{2}[\Gamma_i, \Gamma_j] P_- \right) \mathcal{P}_{\mathcal{R}_n^-} \,, \\
Y_{ij}^- &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r \left(\frac{1}{2}[\Gamma_i, \Gamma_j] P_+ \right) \mathcal{P}_{\mathcal{R}_n^-} \,.
\end{aligned}
\tag{3.1}
$$

The coordinates of the sphere X_i are related to the above by $X_i = X_i^+ + X_i^-$. It is

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also useful to define

$$
\begin{aligned}
X_{ij} &= X^+_{ij} + X^-_{ij}\,,\\
Y_{ij} &= Y^+_{ij} + Y^-_{ij}\,,\\
Y_i &= X^+_i - X^-_i\,,\\
\tilde{X}_{ij} &= X^+_{ij} - X^-_{ij}\,,\\
\tilde{Y}_{ij} &= Y^+_{ij} - Y^-_{ij}\,.
\end{aligned}
\tag{3.2}
$$

Take a state $|s\rangle$ in $\mathcal{R}^+_n$ containing $(n+1)/2$ copies of v_0 and $(n-1)/2$ copies of $a^\dagger_1 v_0$. We find that

$$
\begin{aligned}
\left\langle s|\frac{iX^+_{12}}{n}|s\right\rangle &= -\frac{1}{2}\,,\\
\left\langle s|\frac{iY^+_{12}}{n}|s\right\rangle &= -\frac{1}{2}\,,\\
\left\langle s|\frac{iX^+_{34}}{n}|s\right\rangle &= -\frac{1}{2}\,,\\
\left\langle s|\frac{-iY^+_{34}}{n}|s\right\rangle &= \frac{1}{2}\,.
\end{aligned}
\tag{3.3}
$$

Other expectations values are zero. The algebraic equations obeyed by X^+_{ij} and Y^+_{ij} have a solution with the above numerical values.

These values are stabilized by the rotations L_{12} and L_{34} which generate a $\mathrm{U}(1)\times\mathrm{U}(1)$ subgroup of $\mathrm{SO}(4)$. The $\mathrm{SO}(4)$ acts transitively on the solutions, so the set of solutions is $\mathrm{SO}(4)/\mathrm{U}(1)\times\mathrm{U}(1)$.

4. Summary of relations between generators

There are the sphere-like relations which generalize the defining equation of the three-sphere.

$$
\begin{aligned}
\sum_i X_i X_i &= \frac{(n+1)(n+3)}{2}\,,\\
\sum_{i\neq j} X^+_{ij} X^+_{ij} &= -(n+1)(n+5)\,,\\
\sum_{i\neq j} X^-_{ij} X^-_{ij} &= -(n+1)(n+5)\,,\\
\sum_{i\neq j} Y^+_{ij} Y^+_{ij} &= -(n-1)(n+3)\,,\\
\sum_{i\neq j} Y^-_{ij} Y^-_{ij} &= -(n-1)(n+3)\,.
\end{aligned}
\tag{4.1}
$$

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There are self duality or antiself-duality relations

$$\begin{aligned}
X_{ij}^+ &= \frac{1}{2}\epsilon_{ijkl}X_{kl}^+\,,\\
X_{ij}^- &= -\frac{1}{2}\epsilon_{ijkl}X_{kl}^-\,,\\
Y_{ij}^+ &= -\frac{1}{2}\epsilon_{ijkl}Y_{kl}^-\,,\\
Y_{ij}^- &= \frac{1}{2}\epsilon_{ijkl}Y_{kl}^-\,.
\end{aligned}\tag{4.2}$$

Incidentally, the (anti)self-duality equations show that there are only 3 independent components for X_{ij}^+ and 3 for Y_{ij}^+. We are using conventions $\Gamma_5 = -\Gamma_1\Gamma_2\Gamma_3\Gamma_4$ with $\Gamma_5 v_0 = v_0$. Together with the equations in (4.1) this shows explicitly the origin of the $S^2\times S^2$ in the upper block. Similarly there is the $S^2\times S^2$ in the lower block defined by the X_{ij}^- and Y_{ij}^- variables.

Consider multiplying $X_i^\pm$ with $X_j^\pm$

$$\begin{aligned}
(X_i^+X_j^- - X_j^+X_i^-) + \frac{\epsilon_{ijkl}}{2}(X_k^+X_l^- - X_l^+X_k^-) &= -\frac{(n-1)}{2}\epsilon_{ijkl}X_{kl}^- + 2\left(X_{ij}^- + \frac{\epsilon_{ijkl}}{2}X_{kl}^-\right),\\
(X_i^+X_j^- - X_j^+X_i^-) - \frac{\epsilon_{ijkl}}{2}(X_k^+X_l^- - X_l^+X_k^-) &= -\frac{n+1}{2}\epsilon_{ijkl}Y_{kl}^- + 2\left(X_{ij}^- - \frac{\epsilon_{ijkl}}{2}X_{kl}^-\right).
\end{aligned}\tag{4.3}$$

By converting all $+$ to $-$ and at the same time converting ϵ_{ijkl} to $-\epsilon_{ijkl}$, we can write the folllowing equation:

$$\begin{aligned}
(X_i^-X_j^+ - X_j^-X_i^+) - \frac{\epsilon_{ijkl}}{2}(X_k^-X_l^+ - X_l^-X_k^+) &= \frac{(n-1)}{2}\epsilon_{ijkl}X_{kl}^+ + 2\left(X_{ij}^+ - \frac{\epsilon_{ijkl}}{2}X_{kl}^+\right),\\
(X_i^-X_j^+ - X_j^-X_i^+) + \frac{\epsilon_{ijkl}}{2}(X_k^-X_l^+ - X_l^-X_k^+) &= \frac{n+1}{2}\epsilon_{ijkl}Y_{kl}^+ + 2\left(X_{ij}^+ + \frac{\epsilon_{ijkl}}{2}X_{kl}^+\right).
\end{aligned}\tag{4.4}$$

From these we can write down

$$\begin{aligned}
(X_i^-X_j^+ - X_j^-X_i^+) &= \frac{n}{4}\epsilon_{ijkl}(X_{kl}^+ + Y_{kl}^+) + \frac{1}{4}\epsilon_{ijkl}(-X_{kl}^+ + Y_{kl}^+) + 2X_{ij}^+\,,\\
(X_i^+X_j^- - X_j^+X_i^-) &= -\frac{n}{4}\epsilon_{ijkl}(X_{kl}^- + Y_{kl}^-) + \frac{1}{4}\epsilon_{ijkl}(X_{kl}^- - Y_{kl}^-) + 2X_{ij}^-\,.
\end{aligned}\tag{4.5}$$

By adding the two equations above we get:

$$\begin{aligned}
[X_i, X_j] &= -\frac{n}{4}\epsilon_{ijkl}[(X_{kl}^- - X_{kl}^+) + (Y_{kl}^- - Y_{kl}^+)] - \frac{1}{4}\epsilon_{ijkl}[(X_{kl}^+ - X_{kl}^-) - (Y_{kl}^+ - Y_{kl}^-)] + 2X_{ij}\\
&= \frac{n}{4}\epsilon_{ijkl}(\tilde{X}_{kl} + \tilde{Y}_{kl}) - \frac{1}{4}\epsilon_{ijkl}(\tilde{X}_{kl} - \tilde{Y}_{kl}) + 2X_{ij}\,.
\end{aligned}\tag{4.6}$$

By taking the difference of the two eqs. in (4.5) we find

$$\{X_i^+, X_j^-\} - \{X_j^+, X_i^-\} = -\frac{n}{4}\epsilon_{ijkl}(X_{kl} + Y_{kl}) + \frac{1}{4}\epsilon_{ijkl}(X_{kl} - Y_{kl}) - 2(\tilde{X}_{ij})\,.\tag{4.7}$$

This can be expressed as a commutator of X_i and Y_i.

$$\{X_i, Y_j\} = \frac{n}{4}\epsilon_{ijkl}(X_{kl} + Y_{kl}) - \frac{1}{4}\epsilon_{ijkl}(X_{kl} - Y_{kl}) + 2(\tilde{X}_{ij})\,.\tag{4.8}$$

We of course have relations like

$$
\begin{aligned}
X_i^+ X_j^+ &= 0\,,\\
X_i^- X_j^- &= 0
\end{aligned}
\tag{4.9}
$$

which follow from

$$
\mathcal{P}_{\mathcal{R}_n^+}\mathcal{P}_{\mathcal{R}_n^-} = \mathcal{P}_{\mathcal{R}_n^-}\mathcal{P}_{\mathcal{R}_n^+} = 0\,. \tag{4.10}
$$

Now we consider multiplying $X_{ij}^{\pm}$ and $Y_{ij}^{\pm}$ with $X_k^{\pm}$

$$
\begin{aligned}
X_{kl}^+ X_i^- - X_i^- Y_{kl}^- &= \delta_{il} X_k^- - \delta_{ik} X_l^-\,,\\
X_{kl}^- X_i^+ - X_i^+ Y_{kl}^+ &= \delta_{il} X_k^+ - \delta_{ik} X_l^+\,,\\
Y_{kl}^- X_i^+ - X_i^+ X_{kl}^+ &= \delta_{il} X_k^- - \delta_{ik} X_l^-\,,\\
Y_{kl}^+ X_i^- - X_i^- X_{kl}^- &= \delta_{il} X_k^+ - \delta_{ik} X_l^+\,.
\end{aligned}
\tag{4.11}
$$

Defining $C_{kli}^j \equiv \delta_{il}\delta_{kj} - \delta_{ik}\delta_{jl}$ we can rewrite the above as:

$$
\begin{aligned}
X_{kl}^+ X_i^- - X_i^- Y_{kl}^- &= C_{kli}^j X_j^-\,,\\
X_{kl}^- X_i^+ - X_i^+ Y_{kl}^+ &= C_{kli}^j X_j^+\,,\\
Y_{kl}^- X_i^+ - X_i^+ X_{kl}^+ &= C_{kli}^j X_j^+\,,\\
Y_{kl}^+ X_i^- - X_i^- X_{kl}^- &= C_{kli}^j X_j^-\,.
\end{aligned}
\tag{4.12}
$$

Combining these with the facts:

$$
\begin{aligned}
X_{kl}^+ X_i^+ &= X_i^+ X_{kl}^- = 0\,,\\
X_{kl}^- X_i^- &= X_i^+ X_{kl}^- = 0
\end{aligned}
\tag{4.13}
$$

which follow from (4.10) these imply that $X_{kl}^{\pm} + Y_{kl}^{\pm}$ acts by commutators as generators of SO(4) rotations.

$$
\begin{aligned}
[X_{kl}^+ + Y_{kl}^-, X_i^-] &= C_{kli}^j X_j^-\,,\\
[X_{kl}^- + Y_{kl}^+, X_i^+] &= C_{kli}^j X_j^+\,,\\
[X_{kl}^+ + Y_{kl}^-, X_i^+] &= C_{kli}^j X_j^+\,,\\
[X_{kl}^- + Y_{kl}^+, X_i^-] &= C_{kli}^j X_j^-\,.
\end{aligned}
\tag{4.14}
$$

The differences $X_{kl}^{\pm} - Y_{kl}^{\pm}$ act by anticommutators as a rotation combined with change of X_i^+ to X_i^-:

$$
\begin{aligned}
\{X_{kl}^+ - Y_{kl}^-, X_i^-\} &= C_{kli}^j X_j^-\,,\\
\{X_{kl}^- - Y_{kl}^+, X_i^-\} &= -C_{kli}^j X_j^-\,,\\
\{X_{kl}^- - Y_{kl}^+, X_i^+\} &= C_{kli}^j X_j^+\,,\\
\{X_{kl}^+ - Y_{kl}^-, X_i^+\} &= -C_{kli}^j X_j^+\,.
\end{aligned}
\tag{4.15}
$$

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Recalling from (3.2) that $X_i = X_i^+ + X_i^-$ and $Y_i = X_i^+ - X_i^-$ allows us to write some interesting anti-commutator actions:

$$\begin{aligned} \{X_{kl}^+ - Y_{kl}^-, X_i\} &= -C^j_{kli} Y_j \,, \\ \{X_{kl}^- - Y_{kl}^+, X_i\} &= C^j_{kli} Y_j \,, \\ \{X_{kl}^+ - Y_{kl}^-, Y_i\} &= -C^j_{kli} X_j \,, \\ \{X_{kl}^- - Y_{kl}^+, Y_i\} &= C^j_{kli} X_j \,. \end{aligned} \tag{4.16}$$

It follows that:

$$\begin{aligned} [X_{kl} + Y_{kl}, X_i] &= 2C^j_{kli} X_j \,, \\ [X_{kl} + Y_{kl}, Y_i] &= 2C^j_{kli} Y_j \,, \\ \{X_{kl} - Y_{kl}, X_i\} &= 0 \,, \\ \{X_{kl} - Y_{kl}, Y_i\} &= 0 \,. \end{aligned} \tag{4.17}$$

Recalling from (3.2) that $\tilde{X}_{kl} = X_{kl}^+ - X_{kl}^-$ and $\tilde{Y}_{kl} = Y_{kl}^+ - Y_{kl}^-$ we have

$$\begin{aligned} \{\tilde{X}_{kl} + \tilde{Y}_{kl}, X_i\} &= -2C^j_{kli} Y_j \,, \\ \{\tilde{X}_{kl} + \tilde{Y}_{kl}, Y_i\} &= -2C^j_{kli} X_j \,, \\ [\tilde{X}_{kl} - \tilde{Y}_{kl}, X_i] &= 0 \,, \\ [\tilde{X}_{kl} - \tilde{Y}_{kl}, Y_i] &= 0 \,. \end{aligned} \tag{4.18}$$

So we have interesting anticommutator actions which are SO(4) rotations combined with an $X - Y$ flip.

Finally we need to consider multiplication $X_{ij} X_{kl}$ and $X_{ij} Y_{kl}$ Some of these are familiar from the fuzzy S^4 case:

$$\begin{aligned} [X_{ij}, X_{kl}] &= \delta_{jk} X_{il} + \delta_{il} X_{jk} - \delta_{jl} X_{ik} - \delta_{il} X_{jk} \,, \\ [Y_{ij}, Y_{kl}] &= \delta_{jk} Y_{il} + \delta_{il} Y_{jk} - \delta_{jl} Y_{ik} - \delta_{il} Y_{jk} \,. \end{aligned} \tag{4.19}$$

Further relations are

$$\begin{aligned} [X_{ij}, Y_{kl}] &= 0 \,, \\ X^+_{(ij} Y^+_{kl)} &= -\delta_{jk}(X_i^- X_l^+ + X_l^- X_i^+) - \delta_{il}(X_j^- X_k^+ + X_k^- X_j^+) + \\ &\quad + \delta_{jl}(X_i^- X_k^+ + X_k^- X_i^+) + \delta_{ik}(X_j^- X_l^+ + X_l^- X_j^+) + \\ &\quad + \frac{(n+1)(n+3)}{2}(-\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \mathcal{P}_{\mathcal{R}_n^+} \,, \end{aligned} \tag{4.20}$$

$$X^+_{ij} Y^+_{il} + X^+_{il} Y^+_{ij} = 2(X_j^- X_l^+ + X_l^- X_j^+) - \frac{(n+1)(n+3)}{2} \delta_{jl} \mathcal{P}_{\mathcal{R}_n^+} \,. \tag{4.21}$$

By taking $+$ to $-$ we have

$$\begin{aligned} X^-_{(ij} Y^-_{kl)} &= -\delta_{jk}(X_i^+ X_l^- + X_l^+ X_i^-) - \delta_{il}(X_j^+ X_k^- + X_k^+ X_j^-) + \\ &\quad + \delta_{jl}(X_i^+ X_k^- + X_k^+ X_i^-) + \delta_{ik}(X_j^+ X_l^- + X_l^+ X_j^-) \,, \\ X^-_{ij} Y^-_{il} + X^-_{il} Y^-_{ij} &= 2(X_j^+ X_l^- + X_l^+ X_j^-) - \frac{(n+1)(n+3)}{2} \delta_{jl} \mathcal{P}_{\mathcal{R}_n^-} \,. \end{aligned} \tag{4.22}$$

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This implies

$$
\begin{aligned}
X_{(ij}Y_{kl)} \;=\; \tilde{X}_{(ij},\tilde{Y}_{kl)} = -\delta_{jk}\{X_i,X_l\} - \delta_{il}\{X_j,X_k\} + \delta_{jl}\{X_i,X_k\} + \delta_{ik}\{X_j,X_l\} + \\
+\;\frac{(n+1)(n+3)}{2}(-\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,. \qquad (4.23)
\end{aligned}
$$

A special case of (4.23) is

$$
X_{(ij}Y_{il)} = 2\{X_j,X_l\} - \frac{(n+1)(n+3}{2}\delta_{jl} = -2\{Y_j,Y_l\} - \frac{(n+1)(n+3)}{2}\delta_{jl}\,. \qquad (4.24)
$$

It is worth pointing out that the equations of this section correct some equations in the appendix of [11] and also the remarks [12, section 7.3].

5. Algebra of fuzzy three-sphere at large N

It is useful to define, as in the case of fuzzy even spheres, rescaled variables. A surprising feature will be that, after the standard rescalings, some non commutativity will remain at large n in the fuzzy odd cases.

We can define variables

$$
\begin{aligned}
A^{\pm}_{ij} &= \frac{iX^{\pm}_{ij}}{n}\,, \\
B^{\pm}_{ij} &= \frac{iY^{\pm}_{ij}}{n}\,, \\
C^{\pm}_{i} &= \frac{\sqrt{2}X^{\pm}_{i}}{n}
\end{aligned} \qquad (5.1)
$$

and the following linear combinations are useful:

$$
\begin{aligned}
A_{ij} &= A^{+}_{ij} + A^{-}_{ij}\,, \\
\tilde{A}_{ij} &= A^{+}_{ij} - A^{-}_{ij}\,, \\
B_{ij} &= B^{+}_{ij} + B^{-}_{ij}\,, \\
\tilde{B}_{ij} &= B^{+}_{ij} - B^{-}_{ij}\,, \\
C_{i} &= (C^{+}_{i} + C^{-}_{i})\,, \\
\tilde{C}_{i} &= i(C^{+}_{i} - C^{-}_{i})\,.
\end{aligned} \qquad (5.2)
$$

The sphere-like relations are of the form:

$$
\begin{aligned}
\sum_{i\neq j}(A^{\pm}_{ij})^2 &= \mathcal{P}_{\mathcal{R}^{\pm}_n}\,, \qquad \sum_{i\neq j}(B^{\pm}_{ij})^2 = \mathcal{P}_{\mathcal{R}^{\pm}_n}\,, \\
\sum_{i\neq j}(A_{ij})^2 &= 1\,, \qquad \sum_{i\neq j}(B_{ij})^2 = 1\,, \\
\sum_{i} C_i^2 &= 1\,, \qquad \sum_{i}\tilde{C}_i^2 = 1\,.
\end{aligned} \qquad (5.3)
$$

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We have the following vanishing commutators and anticomutators:

$$
\begin{aligned}
[A, A] &= 0\,, \\
[B, B] &= 0\,, \\
[A, C] = [A, \tilde{C}] &= 0\,, \\
[\tilde{A}+\tilde{B}, C] &= [\tilde{A}+\tilde{B}, \tilde{C}] = 0\,, \\
\{\tilde{A}-\tilde{B}, C\} &= \{\tilde{A}-\tilde{B}, \tilde{C}\} = 0\,.
\end{aligned} \tag{5.4}
$$

There are also some interesting commutators of $C, \tilde{C}$ which survive in the large-N limit

$$
\begin{aligned}
\left(C_i^- C_j^+ - C_j^- C_i^+\right) &= \frac{-i}{2}\epsilon_{ijkl}\left(A_{kl}^+ + B_{kl}^+\right), \\
\left(C_i^+ C_j^- - C_j^+ C_i^-\right) &= \frac{i}{2}\epsilon_{ijkl}\left(A_{kl}^- + B_{kl}^-\right).
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
[C_i, C_j] &= \frac{-i}{2}\epsilon_{ijkl}\left(\tilde{A}_{kl} + \tilde{B}_{kl}\right), \\
\{C_i, \tilde{C}_j\} &= \frac{-1}{2}\epsilon_{ijkl}\left(A_{kl} + B_{kl}\right), \\
A_{(ij}B_{kl)} &= -\{\tilde{A}_{ij}, \tilde{B}_{kl}\} \\
&= \frac{\delta_{jk}}{2}\{C_i, C_l\} + \frac{\delta_{il}}{2}\{C_j, C_k\} - \frac{\delta_{jl}}{2}\{C_i, C_k\} - \\
&\quad - \frac{\delta_{ik}}{2}\{C_j, C_l\} + \frac{1}{3}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
&= +\frac{\delta_{jk}}{2}\left\{\tilde{C}_i, \tilde{C}_l\right\} + \frac{\delta_{il}}{2}\left\{\tilde{C}_j, \tilde{C}_k\right\} - \frac{\delta_{jl}}{2}\left\{\tilde{C}_i, \tilde{C}_k\right\} - \\
&\quad - \frac{\delta_{ik}}{2}\{\tilde{C}_j, \tilde{C}_l\} + \frac{1}{3}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
A_{(ij}B_{il)} &= -\{C_j, C_l\} + \frac{\delta_{jl}}{2}\,.
\end{aligned} \tag{5.6}
$$

So A, B should be thought of as coordinates and C could perhaps be realized as derivatives. This is compatible with the observation based on group theory, that the diagonal blocks are describing functions on two copies of $S^2 \times S^2$, while the off-diagonal are some sections of a bundle. The fact that within each diagonal block $\mathrm{End}(\mathcal{R}_n^+)$ or $\mathrm{End}(\mathcal{R}_n^-)$ the algebra becomes commutative in the large-n limit, whereas there is non commutativity in $\mathrm{End}(\mathcal{R}_n)$ suggests that some sort of U(2) non-abelian structure is present. We will return to this in section 7.

6. Some matrix actions which admit the fuzzy three-sphere as solution

It was observed in [15] that the fuzzy two-sphere and four sphere are solutions to matrix brane actions which have a mass term. Such matrix models in relation to fuzzy spheres were also discussed in [23]. Mass terms also appear in the BMN matrix model and the fuzzy two-sphere appears as a supersymmetric solution of the BMN matrix model [6]. It is natural to ask if a simple matrix Model with mass term can accommodate the fuzzy three sphere as a solution. In the case of the fuzzy even spheres the anlog of (4.6) is much simpler, since it only contains the analog of the last term on the r.h.s. of (4.6). Further, the commutator $[X_i, X_{ij}]$ is proportional to X_j in the even case, unlike the more complicated analog (4.17) which involves both X and Y variables. These properties of the fuzzy even spheres are adequate to prove they are solutions of a matrix model with X_i in the vector of the SO symmetry group and having a mass term. The form of the equations for the fuzzy three-sphere do not allow it to be a solution of such a matrix model. It is a very interesting problem to try to find *any* physical matrix model where the fuzzy three-sphere (or the fuzzy five sphere discussed later in this paper) is a solution. It would be even more fascinating to find these general fuzzy spheres as supersymmetry-perserving solution of any physical matrix model. A physical matrix model would be one that comes up as an action for branes (parallel or intersecting) or as a dual to some M-theory background. In the light of the difficulty of constructing a transverse five-brane from the BFSS matrix model, finding such a fuzzy five sphere as a physical matrix model solution (or proving it is not possible) would give important information about the relation between M-theory and large-N matrix systems. We will not solve this problem in this paper but we will exhibit some toy matrix models which do have the fuzzy three sphere as a solution, and we will observe some qualititative similarities between these toy matrix models and some others that have appeared in recent literature.

Given the relations we wrote down in (4.6) and (4.7) it is easy to obtain some matrix Actions which admit as solutions the matrices of the fuzzy three-sphere. Consider a matrix action with a symmetry group which contains SO(4) and consider fields transforming in the vector of SO(4), labelled Φ_i as in the BFSS [24] or IKKT [25, 26] matrix models. Suppose, unlike these models there is an additional matrix variable transforming in the antisymmetric of SO(4) which we will denote P_{ij}. Suppose the action is

$$\int \mathrm{TR}\left([\Phi_i, \Phi_j] - P_{ij}\right)^2 . \tag{6.1}$$

The variation of this action with respect to Φ_i or P_{ij} will be proportional to

$$[\Phi_i, \Phi_j] - P_{ij} . \tag{6.2}$$

From (4.6) it follows that if we set

$$\begin{aligned} \Phi_i &= X_i \,, \\ P_{ij} &= \frac{n}{4}\epsilon_{ijkl}(\tilde{X}_{kl} + \tilde{Y}_{kl}) - \frac{1}{4}\epsilon_{ijkl}\left(\tilde{X}_{kl} - \tilde{Y}_{kl}\right) + 2X_{ij} \,, \end{aligned} \tag{6.3}$$

where $X_i, X_{kl}, Y_{kl}, \tilde{X}_{kl}, \tilde{Y}_{kl}$ are defined in (3.1) and (3.2), we will have a solution to the action (6.1).

Another way to construct an action which is solved by the matrices of section 3 is to start with variables Φ_i and $\tilde{\Phi}_i$ and antisymmetric matrix variables Q_{ij} which have an action

$$\int \mathrm{TR}\left(\left\{\Phi_i, \tilde{\Phi}_j\right\} - Q_{ij}\right)^2 . \tag{6.4}$$

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In this case if we use

$$\begin{aligned}
\Phi_i &= X_i\,, \\
\tilde{\Phi}_i &= Y_i\,, \\
Q_{ij} &= \frac{n}{4}\epsilon_{ijkl}(X_{kl}+Y_{kl}) - \frac{1}{4}\epsilon_{ijkl}(X_{kl}-Y_{kl}) + 2(\tilde{X}_{ij})\,.
\end{aligned} \tag{6.5}$$

It is interesting that in recent literature matrix actions have been considered which involve matrices transforming in the vector of the SO group as well as other antisymmetric tensors [27]–[30]. It would be interesting to see if the fuzzy odd sphere solutions of model actions of the form (6.1) could also be found for actions involved in the above papers. Perhaps relating terms of the form in (6.1) to terms in these actions could be useful. The role of P_{ij} could also be played by composite fields such as $\bar{\psi}\Gamma_{ij}\psi$.

7. Fluctuating fields around solution

7.1 Physics on $S^2 \times S^2$

Useful information about the physics of a solution defined by the X_i matrices is obtained by considering fluctuations around such a solution. In the even sphere case the action for the fluctuations is a U(1) theory on $SO(2k+1)/U(k)$ with $SO(2k+1)$ symmetry. The existence of a hidden higher dimensional coset in the odd sphere allows a somewhat analogous result.

Imagine we find a solution to matrix theory which uses these matrices

$$\begin{aligned}
\Phi_i &= X_i\,, \\
\Phi_a &= 0\,.
\end{aligned} \tag{7.1}$$

The Φ_i transform as a vector of the SO(4) symmetry group of the fuzzy three sphere, and we call them parallel scalars. The Φ_a are invariant under that symmetry group and are called transverse scalars.

Consider the fluctuations

$$\begin{aligned}
\Phi_i &= X_i + \phi_i^+(A^+,B^+) + \phi_i^-(A^-,B^-) + \phi_{ij}^+(A^+,B^+)C_i^- + \phi_{ij}^-(A^-,B^-)C_j^+\,, \\
\Phi_a &= \phi_a^+(A^+,B^+) + \phi_a^-(A^-,B^-) + \phi_{ai}^+(A^+,B^+)C_i^- + \phi_{ai}^-(A^-,B^-)C_i^+\,.
\end{aligned} \tag{7.2}$$

Each ϕ is a field living on $S^2 \times S^2$. Notice that the transverse scalars contain components which transform as vectors under the parallel SO(4). The parallel scalars give rise to fields $\phi_{ij}^{\pm}$ which is a two-index tensor which can be reduced to symmetric traceless part, antisymmetric part and a scalar. By expanding a matrix action which is solved by the fuzzy three-sphere matrices, around the solution, we would get a field theory, in analogy to the analogous results for fuzzy even spheres [13, 15, 31].

We outline some features of the field theory on $S^2 \times S^2$ that results from these fluctuations. The kinetic term of Φ_a is

$$\int dt\ \mathrm{TR}\,(\partial_t\Phi_a\partial_t\Phi_a)\,. \tag{7.3}$$

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Using the expansion for Φ_a in (7.2), we will get, from the diagonal terms,

$$\int dt dA^+ dB^+ \left(\partial_t \phi_a^+(A^+, B^+)\right)^2 + \int dt dA^- dB^- \left(\partial_t \phi_a^-(A^-, B^-)\right)^2 =$$
$$= \int dt dA dB (\partial_t \phi_a^+(A, B))^2 + (\partial_t \phi_a^-(A, B))^2 \tag{7.4}$$

$dA^+dB^+ = dA^-dB^-$ is an SO(4) invariant measure for $S^2 \times S^2$. These diagonal terms coming from the large-n limit of End($\mathcal{R}_n^+$) and End($\mathcal{R}_n^-$). From squaring the off diagonal terms, $\phi_{ai}^+(A^+, B^+)C_i^- + \phi_{ai}^-(A^-, B^-)C_i^+$ we obtain terms including

$$\int dt\, TR_{\mathcal{R}_n^+}(\partial_t \phi_{ai}^+(A^+, B^+)C_i^- \partial_t \phi_{ai}^-(A^-, B^-)C_j^+) \,. \tag{7.5}$$

After rescalings of (5.1 the relations in (4.12) allow us to show that polynomials in A^-, B^- can be pulled to the left of C_i^- at the cost of converting the pair (A^-, B^-) to (B^+, A^+) (not the switch). There are extra terms coming from the r.h.s. of (4.12) which can be ignored in the leading large-n limit. The terms in (7.5) can then be written as

$$\int dt\, TR_{\mathcal{R}_n^+}(\partial_t \phi_{ai}^+(A^+, B^+)\partial_t \phi_{aj}^-(B^+, A^+)C_i^- C_j^+) \,. \tag{7.6}$$

It is useful to write this as a product of parts symmetric in (ij) and a part antisymmetric in these indices. The symmetric part is simplified using (4.20) to replace

$$(C_i^- C_j^+ + C_j^- C_i^+) = \frac{1}{2}\delta_{ij} - A_{ki}^+ B_{kj}^+ - A_{kj}^+ B_{ki}^+ \equiv G_{ij} \,. \tag{7.7}$$

The antisymmetric part is simplified using (4.5)

$$\left(C_i^- C_j^+ - C_j^- C_i^+\right) = \frac{-i}{2}\epsilon_{ijkl}(A_{kl}^+ + B_{kl}^+) \equiv H_{ij} \,. \tag{7.8}$$

We have defined in (7.7) and (7.8) a symmetric tensor and an antisymmetric tensor living on $S^2 \times S^2$. After converting the trace to an integral using SO(4) invariance, we can write the kinetic terms in (7.5) as

$$\int dt \int dA dB \partial_t \phi_{ai}^+(A, B)\partial_t \phi_{aj}^-(B, A)(G_{ij} + H_{ij}) \,. \tag{7.9}$$

Analogous to the term in (7.5) there is a trace over $\mathcal{R}_n^-$ which leads by similar steps as above to

$$\int dt \int dA dB \partial_t \phi_{ai}^-(A, B)\partial_t \phi_{aj}^+(B, A)(G_{ij} - H_{ij}) \,. \tag{7.10}$$

Although H contains i the action is hermitean since hermitean conjugation converts ϕ^+ to ϕ^-.

The above arguments illustrate the use of the algebraic relations of section 4 in deriving the action for fluctuations. It appears that there is some U(2) symmetry which mixes the $\mathcal{R}_n^+$ and the $\mathcal{R}_n^-$ but which is broken to U(1)×U(1). We leave it to the future to elucidate all the symmetries of such an action. A further interesting problem is to relate this theory on $S^2 \times S^2$ to some field theory on S^3. In the fuzzy even sphere case there was a U(1) theory on the higher dimensional geometry and a non-abelian theory on the sphere itself [13].

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We would expect a generalization of that correspondence to yield some field theory on the sphere S^3. The counting of degrees of freedom in section 9 shows that it cannot be a standard non-abelian theory. The discussion in section 9 will also show that the analogous field theory on the coset $\mathrm{SO}(6)/\mathrm{U}(2) \times \mathrm{U}(1)$ might be more easily related to a field theory on the five sphere.

8. The fuzzy five-sphere

8.1 Higher dimensional geometry from stabilizer group

A guess is that $\mathrm{End}(\mathcal{R}_n^+)$ related to $\mathrm{SO}(2k)/\mathrm{U}(k-1) \times \mathrm{U}(1)$. This works in the case of $k = 2$, and gives the right counting of degrees of freedom, i.e. N^2 scales like n^{10} while quadratic expressions like X_i^2 scale like n^2. This means we should expect a ten-dimensional space which is indeed the dimension of the above coset for $k = 3$. Here we will develop an argument similar to section 3.3 based on the expectation values of generators of the matrix algebra evaluated in a state in $\mathcal{R}_n$. We pick a state of the form $v_0 \otimes \cdots \otimes v_0 \otimes a_1^\dagger v_0 \cdots a_1^\dagger v_0$, where there are $(n+1)/2$ copies of v_0 and $(n-1)/2$ copies of $a_1^\dagger v_0$. We sum over different ways of embedding the copies v_0 and the $a_1^\dagger v_0$ in $V^{\otimes n}$ in order to make sure we have a state in $\mathrm{Sym}(V^{\otimes n})$.

Considering normalized quantities analogous to (5.1), we have in the large-n limit

$$
\begin{aligned}
\langle s|A_{12}^+|s\rangle &= 1\,, \\
\langle s|A_{34}^+|s\rangle &= \langle s|A_{56}^+|s\rangle = 1\,, \\
\langle s|C_1^2|s\rangle &= \frac{1}{2}\,, \\
\langle s|C_2^2|s\rangle &= \frac{1}{2}\,, \\
\langle s|C_1C_2 + C_2C_1|s\rangle &= 0\,, \\
\langle s|C_1C_3 + C_3C_1|s\rangle &= 0\ldots\,, \\
\langle s|B_{12}^+|s\rangle &= -1\,, \\
\langle s|B_{34}^+|s\rangle &= \langle s|B_{56}^+|s\rangle = 1\,.
\end{aligned}
\tag{8.1}
$$

Other A_{ij} and B_{ij} have zero expectation value. The above variables are defined in the same way as for the fuzzy three-sphere. These expectation values in (8.1) are only preseved by $\mathrm{U}(1) \times \mathrm{U}(2)$. This means that the complete set of variables describing the matrix algebra $\mathrm{End}(\mathcal{R}_n^+)$ in the large-n limit have a solution which is stabilized by the subgroup $\mathrm{U}(1)\times\mathrm{U}(2)$. The full set of solutions, obtained by action of $\mathrm{SO}(6)$ on the irrep $\mathcal{R}_n^+$, is acted on transitively by the $\mathrm{SO}(6)$. Hence the set of solutions is the coset $\mathrm{SO}(6)/\,\mathrm{U}(2) \times \mathrm{U}(1)$. The $\mathrm{U}(1)$ is generated by rotations in the 12 direction. In the 3546 block there is a $\mathrm{U}(2)$ subgroup which preserves the expectation values. Consider $\langle X_1^2\rangle = \langle X_2^2\rangle = 1$. These are matrices of the form

$$
\begin{pmatrix} P & Q \\ -Q & P \end{pmatrix},
\tag{8.2}
$$

where P is real antisymetric and Q is real symmtric.

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8.2 Group theory proof: harmonics of SO(6)/ U(2) × U(1) from matrix realization

In the next two sections, we develop the arguments analogous to sections 3.1 and 3.2, now for the case of the fuzzy five-sphere. We recall from [12] the list of representations of SO(6) which appear in $\mathrm{End}(\mathcal{R}_n^+)$ at large n. There are self-dual representations associated with highest weights $\vec{\lambda} = (p_1 + p_2 + p_3, p_1 + p_2, p_1)$, and antiself-dual reps with highest weights $\vec{\lambda} = (p_1+p_2+p_3, p_1+p_2, -p_1)$, where p_1, p_2, p_3 are positive. The associated Young diagrams have row lengths $\vec{r} = (p_1 + p_2 + p_3, p_1 + p_2, p_1)$ Unlike the case of the fuzzy three sphere, we now have representations with multiplicity more than 2. The multiplicity is zero unless the following conditions are satisfied: $p_1 + p_3$ is even, if p_1 is positive, $p_3 \geq p_1$ and if p_1 is negative, $p_3 \geq |p_1|$. Then the multiplicity is

$$m(p_1, p_2, p_3) = p_2 + 1\,. \tag{8.3}$$

The multiplicity arises because of the multiple ways of writing p_2 as a sum of two non-negative integers p_2^+ and p_2^-. Different choices of p_2^+ and p_2^- lead to different operators in $\mathrm{End}(\mathcal{R}_n^+)$ which transform according to the same representation of SO(6).

The space of functions on SO(6)/U(2) × U(1) is the SO(6) representation induced from the trivial rep. of U(2) × U(1). By Frobenius duality, multiplicity of an SO(6) irrep. in the induced rep is the same as the multiplicity of the trivial rep. of U(2) × U(1) in that SO(6) representation [20]. We will compare, therefore, the multiplicity of the trivial rep. of U(1) × U(2) in the restriction of any given SO(6) irreducible representation with the number of times that irrep. appears in $\mathrm{End}(\mathcal{R}_n^+)$. We expect they are equal.

We saw in (8.2) that the multiplicities in $\mathrm{End}(\mathcal{R}_n^+)$ are zero unless p_1+p_3 is even. This can be shown easily to be a property of the restriction multiplicities as well, from facts about the root lattice of SO(6) [32]. We show that unless $p_1 + p_3$ is an even integer, the multiplicity of appearance of the trivial rep. of U(2) × U(1) is zero. A necessary condition for the trivial rep. to appear is that the multiplicity of the zero weight vector is non zero. Unless $p_1 + p_3$ is even this multiplicity is actually zero.

Indeed, the positive roots are $L_1 \pm L_2$, $L_2 \pm L_3$, $L_1 \pm L_3$ [32]. The highest weight of the irrep. characterized by $\vec{\lambda} = (p_1 + p_2 + p_3, p_1 + p_2, p_1)$ is $(p_1 + p_2 + p_3)(L_1) + (p_1 + p_2)(L_1 + L_2) + (p_1)(L_1 + L_2 + L_3)$. If any state of weight zero exists in the rep. it must be obtained by subtracting from the highest weight a positive root. So we need to write

$$\vec{\lambda} = a_1(L_1{+}L_2){+}a_2(L_1{-}L_2){+}a_3(L_2{+}L_3){+}a_4(L_2{-}L_3){+}a_5(L_1{+}L_3){+}a_6(L_1{-}L_3)\,. \tag{8.4}$$

If we add up the components of $\vec{\lambda}$ we get $2p_2 + 3p_1 + p_3$ which has to equal $2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6$. Since the a_i are positive integers, it is clear that this is only possible if $p_1 + p_3$ is even.

In the case where $p_1 + p_3$ is even but $p_3 < p_1$ the multiplicity of the identity rep. of U(1) × U(2) is zero. The identity rep. is characterized by having a state of weight zero, which is annihilated by the raising or lowering operators of the U(2). In the case $p_1 + p_3$ odd, the multiplicity of the zero weight vector is zero, so it is clear that there is no trivial rep. In the case where $p_1 + p_3$ is even but $p_3 < p_1$, the multiplicity of the zero weight vector is non zero, but it belongs to a higher rep. of U(1) × U(2).

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We need to recall some details about the embedding of $\mathrm{U}(1)\times\mathrm{U}(2)$. Let H_1, H_2, H_3 be the generators of the Cartan of SO(6). $H_1 = L_{12}, H_2 = L_{34}, H_3 = L_{56}$. The U(1) appearing as a factor in the subgroup is generated by H_1. The abelian U(1) of the U(2) is generated by $H_2 - H_3$. The Cartan of the SU(2) subgroup of the U(2) is $H_2 + H_3$. The trivial rep. of $\mathrm{U}(1)\times\mathrm{U}(2)$ has $(H_1, H_2, H_3) = (0, 0, 0)$. Non-trivial reps. of this subgroup which also contain the zero vector have highest weights of the form $(0, k, -k)$. Let $m(0, k, -k)$ be the multiplicity of the state of weight $(0, k, -k)$. Let $m_h(0, k, -k)$ be the multiplicity of $(0, k, -k)$ as a highest weight. For a fixed rep. of SO(6) let $(0, K, -K)$ be the highest $\mathrm{U}(1)\times\mathrm{U}(2)$ weight which appears in the restriction to $\mathrm{U}(1)\times\mathrm{U}(2)$. The problem of finding the multiplicity of the trivial rep. is then reduced to calculating multiplicities of such states. Indeed

$$\begin{aligned}
m_h(0,0,0) &= m(0,0,0) - m_h(0,0,0) - m_h(0,1,-1) - \cdots m_h(0,K,-K)\,,\\
m_h(0,1,-1) &= m(0,1,-1) - m_h(0,2,-2) - \cdots m_h(0,K,-K)\,,\\
m_h(0,2,-2) &= m(0,2,-2) - m_h(0,3,-3) - \cdots m_h(0,K,-K)\,,\\
&\;\vdots\\
m_h(0,K,-K) &= m(0,K,-K)\,.
\end{aligned} \tag{8.5}$$

From these equations it follows that

$$m_h(0,0,0) = m(0,0,0) - m(0,1,-1)\,. \tag{8.6}$$

In table 1 below we give the relevant multiplicities for some of irreps. of SO(6) and we calculate $m_h(0, 0, 0)$. The multiplicity obtained from the matrix decomposition has been expressed in the final column using a θ function to express the fact that it is non zero only when $p_3 > |p_1|$ The multiplicities of the weight vectors are calculated on Maple using the Weyl character formula for SO(6) (see for example [32]). We observe that the multiplicity calculated using the character formula is equal to that we expect from the matrix Decomposition, as shown by the agreement of the last two columns of table 1.

We have also performed the above checks when p_1 is negative, i.e. the reps are anti-self-dual, and we again get agreement between the matrix decomposition multiplicities and those obtained from counting weights. It would be interesting to develop this proof using characters into an analytic proof but we will leave that to the enthusiastic reader, and hope the remaining readers will be content with the agreement demonstrated in table 1.

8.3 $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$ as sections

We have matched the matrices which map $\mathcal{R}_n^+$ to $\mathcal{R}_n^+$, i.e. endomorphisms $\mathrm{End}(\mathcal{R}_n^+, \mathcal{R}_n^+)$ to functions on the coset $\mathrm{SO}(6)/\mathrm{U}(2)\times\mathrm{U}(1)$. The matrices $\mathrm{End}(\mathcal{R}_n^-, \mathcal{R}_n^-)$ have the same decomposition into SO(6) representations as $\mathrm{End}(\mathcal{R}_n^+, \mathcal{R}_n^+)$. The matrices which map $\mathcal{R}_n^+$ to $\mathcal{R}^-$, or homomorphisms $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$ have a different SO(6) decomposition. Here we will show that the SO(6) decomposition of $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$ matches that of sections of a certain bundle over $\mathrm{SO}(6)/\mathrm{U}(2)\times\mathrm{U}(1)$.

Sections of bundles over a coset G/H are induced representations. They are specified by choosing a representation R_H of the subgroup H constructing the representation of G

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$\vec{\lambda}$	$m(0,0,0)$	$m(0,1,-1)$	$m_h(0,0,0)$	$(p_2+1)\Theta(p_3-p_1)$
(2,0,0)	2	1	1	1
(4,0,0)	3	2	1	1
(6,0,0)	4	3	1	1
(8,0,0)	5	4	1	1
(1,1, 0)	3	1	2	2
(3,1, 0)	7	5	2	2
(5,1, 0)	11	9	2	2
(2,2, 0)	6	3	3	3
(4,2, 0)	15	12	3	3
(6,2, 0)	24	21	3	3
(3,3, 0)	10	6	4	4
(5,3, 0)	26	22	4	4
(7,3, 0)	42	38	4	4
(4,4, 0)	15	10	5	5
(6,4, 0)	40	35	5	5
(5,5, 0)	21	15	6	6
(2,1, 1)	3	2	1	1
(4,1, 1)	6	5	1	1
(6,1, 1)	9	8	1	1
(3,2, 1)	8	6	2	2
(5,2, 1)	16	14	2	2
(7,2, 1)	24	22	2	2
(4,3, 1)	15	12	3	3
(6,3, 1)	30	27	3	3
(5,4, 1)	24	20	4	4
(4,2, 2)	6	5	1	1
(6,2, 2)	11	10	1	1
(5,3, 2)	15	13	2	2
(4,4, 2)	6	6	0	0
(4,3, 3)	3	3	0	0
(6,3, 3)	10	9	1	1
(7,4, 3)	24	22	1	1

Table 1: Comparison of restriction multiplicities of SO(6) to trivial representation of $\mathrm{U}(1)\times\mathrm{U}(2)$.

induced from it. In the discussion above, R_H was the trivial bundle. By the Frobenius reciprocity theorem, the multiplicity of an irreducible representation of R_G of G in the induced representation is equal to the multiplicity of R_H in the restriction of R_G to H.

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Sections of a non-trivial bundle are obtained by chooosing a non-trivial representation of H. Here we show that $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$ matches the representation of SO(6) induced from $(1,0,0)$ of $U(1) \times U(2)$. The first integer here is the U(1) charge and the last two integers label the representation of U(2). $\mathrm{Hom}(\mathcal{R}_n^-, \mathcal{R}_n^+)$ matches the representation of SO(6) induced from $(-1,0,0)$ of $U(1) \times U(2)$.

We will give the evidence in the case of $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$. Let us recall from [12] the multiplicities of irreps. of SO(6) appearing in $\mathrm{Hom}(\mathcal{R}_n^+, \mathcal{R}_n^-)$. For positive integers p_1, p_2, p_3 there are self-dual representations $(\lambda_1, \lambda_2, \lambda_3) = (p_1+p_2+p_3, p_1+p_2, p_1)$. The multiplicities are $m(\lambda) = p_2 + 1$ if $p_1 + p_3$ is odd and $p_3 \geq p_1 - 1$ If these conditions on p_1 and p_3 are not satisfied, then the multiplicity is zero. At finite n there is an upper limit $p_1 + p_2 + p_3 \leq n$, but as $n \to \infty$ there is no upper limit. There are also anti-selfdual representations labelled by $(\lambda_1, \lambda_2, \lambda_3) = (p_1+p_2+p_3, p_1+p_2, -p_1)$, so that λ_3 is negative, while p_1, p_2, p_3 are still positive. Again the condition for a non-zero multiplicity is that p_1+p_3 is odd, and that $p_3 \geq p_1+1$. When these conditions are satisfied, the multiplicity is given by $m(\vec{\lambda}) = p_2 + 1$. These multiplicities from the matrix deomposition have to agree with restriction multiplicities, i.e. the number of times $(1,0,0)$ appears as a $U(2) \times U(1)$ highest weight for any given highest weight $\vec{\lambda}$ of SO(6).

The condition $p_1 + p_3$ be odd is easy to prove (just as the evenness of $p_1 + p_3$ is easy to prove in the case of restriction multiplicities for the trivial rep. of SO(6)). It simply follows from the fact that if p_1+p_3 is even, the corresponding highest weight differs from the vector $(1,0,0)$ by an element which is not in the root lattice. This means that the multiplicity of $(1,0,0)$ is zero. In the case where $p_1 + p_3$ is odd but the inequalities between p_1 and p_3 are not respected, the multiplicity of the weight vector $(1,0,0)$ is non zero, but its multiplicity as a highest weight of $U(1) \times U(2)$ is zero. The multiplicity of $(1,0,0)$ as a highest weight of $U(1) \times U(2)$, which we denote as $m_h(1,0,0)$, is related to the multiplicities of weights of the form $(1,0,0), (1,1,-1), \ldots\ (1,K,-K)$, where $(1,K,-K)$ is the highest weight of this form appearing in the irrep. specified by the highest weight $\vec{\lambda}$. $m(1,0,0)$ and $m(1,1,-1)$ by

$$\begin{aligned} m_h(1,0,0) &= m(1,0,0) - (m_h(1,1,-1) + \cdots + m_h(1,K,-K))\,, \\ m_h(1,1,-1) &= m(1,1,-1) - (m_h(1,1,-1) + \cdots m_h(1,K,-K))\,, \\ &\vdots \\ m_h(1,2,-2) &= m(1,2,-2) - (m_h(1,2,-2) + \cdots m(1,K,-K))\,, \\ m_h(1,K,-K) &= m(1,K,-K) \end{aligned} \tag{8.7, 8.8}$$

which leads to a simple equation for $m_h(1,0,0)$ in terms of $m(1,0,0)$ and $m(1,1,-1)$.

$$m_h(1,0,0) = m(1,0,0) - m(1,1,-1)\,. \tag{8.9}$$

Table 2 gives a counting of weight multiplicities performed with Maple using the Weyl character formula for SO(6) and shows that the counting of multiplicities of the $U(2) \times U(1)$ irrep. obtained by restriction of SO(6) agrees with that obtained from analysing the SO(6) content of $\mathrm{Hom}(\mathcal{R}_n^+ . \mathcal{R}_n^-)$.

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$\vec{\lambda}$	$m(1,0,0)$	$m(1,1,-1)$	$m_h(1,0,0)$	$(p_2+1)\Theta(p_3-p_1+1)$
(1,0,0)	1	0	1	1
(3,0,0)	2	1	1	1
(5,0,0)	3	2	1	1
(7,0,0)	4	3	1	1
(2,1, 0)	4	2	2	2
(4,1, 0)	8	6	2	2
(6 ,1, 0)	12	10	2	2
(8 ,1, 0)	16	14	2	2
(3,2, 0)	9	6	3	3
(5,2, 0)	18	15	3	3
(7,2, 0)	27	24	3	3
(4,3, 0)	16	12	4	4
(6,3, 0)	32	28	4	4
(5,4, 0)	25	20	5	5
(3,1, 1)	4	3	1	1
(5,1, 1)	7	6	1	1
(7,1, 1)	10	9	1	1
(5,2, 2)	8	7	1	1
(7,2, 2)	13	12	1	1
(4,3, 2)	8	6	2	2
(6,3, 2)	20	18	2	2
(3,3, 3)	1	1	0	0
(5,3, 3)	6	5	1	1
(5,4, 4)	3	3	0	0
(7,4, 4)	10	9	1	1

Table 2: Comparison of restriction multiplicities of SO(6) to a non-trivial representation of U(1) × U(2).

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9. Some remarks on geometry and combinatorics of fuzzy odd spheres

9.1 geometry of fuzzy S^{2k}

The coset $\mathrm{SO}(2k)/\mathrm{U}(k-1)\times\mathrm{U}(1)$ can be seen to admit the following bundle structure,

$$\begin{array}{ccc} \dfrac{\mathrm{SO}(2k)}{\mathrm{U}(k-1)\times\mathrm{U}(1)} & \longleftarrow & \dfrac{\mathrm{U}(k)}{\mathrm{U}(k-1)\times\mathrm{U}(1)} \\ \downarrow & & \\ \dfrac{\mathrm{SO}(2k)}{\mathrm{U}(k)} & & \end{array} \tag{9.1}$$

using the general bundle structure $H/K \to G/K \to G/H$ which exists whwnever we have subgroups $K \subset H \subset G$. The base is a hermitean symmetric space (incidentally one of the kind that comes up as the fibre of the bundle structure exploited in [13] for fuzzy even spheres). The fibre here is actually a coset description of CP^{k-1}, also a symmetric space. In the case $k=2$, the base and fibre are both S^2. In the case $k=3$, the base is CP^3 and the fibre CP^2. In the case $k=4$, the fibre is CP^4 and the base is $\mathrm{SO}(8)/\mathrm{U}(4)$.

In the case $k=2$, the base S^2 is of lower dimension than the sphere S^3 we started with. For higher k the base of the above fibration is always of higher dimension than the odd sphere S^{2k-1}. The case $k=3$ of the fuzzy five sphere is rather special in that the CP^3 is simply related to the S^5 by a U(1) quotient, $S^5 = CP^3/\mathrm{U}(1)$, a fact which found a recent physical application in [33] for example. In [13] the bundle structure of the higher dimensional geometry was used to obtain a non-abelian theory on the sphere. It is possible that the special relation of CP^3 to S^5 will allow the matrix algebra related to fuzzy five-sphere to be related to field theory on five-sphere, and hence to brane physics of spherical branes, more easily than the other cases. It is intriguing that the physics of type-IIA theory which contains both supersymmetric zero-branes and NS-5-branes might also be taken to suggest, using a simple-minded asumption that zero-branes should in some context blow-up into NS-5, that fuzzy five-spheres would be easier than other odd spheres.

Another possibly useful bundle structure is

$$\begin{array}{ccc} \dfrac{\mathrm{SO}(2k)}{\mathrm{U}(k-1)\times\mathrm{U}(1)} & \longleftarrow & \dfrac{\mathrm{SO}(2k-2)\times\mathrm{U}(1)}{\mathrm{U}(k-1)\times\mathrm{U}(1)} = \dfrac{\mathrm{SO}(2k-2)}{\mathrm{U}(k-1)} \\ \downarrow & & \\ \dfrac{\mathrm{SO}(2k)}{\mathrm{U}(1)\times\mathrm{SO}(2k-2)}\,. & & \end{array} \tag{9.2}$$

9.2 Projecting to S^{2k-1} and possible role for $S^{2k-1}/Z_2 = RP^{2k-1}$

In the case of the fuzzy even sphere, there is a simple way to project the matrix algebra onto the $\mathrm{SO}(2k+1)$ reps which are symmetric traceless, and which approach the algebra of functions on a sphere at large n. This gives rise to a non-associative algebra at finite n [12, 13]. We can apply a projection to get a space of functions which is the same as the space of functions on S^{2k-1}. This requires the off diagonal components. If we only keep matrices which transform as symmetric traceless irreps. of $\mathrm{SO}(2k-1)$ and and which are invariant under conjugation by a permutation which switches positive chirality with negative, this will give us the desired representations. This projects out the Y_i, while keeping X_i and their symmetric products. It also kills off $X^{\pm}_{ij}, Y^{\pm}_{ij}$

If we look at the upper block alone $\mathrm{End}(\mathcal{R}^+_n)$ and project out antisymmetric representations we are left with the algebra of functions on $RP^{2k-1} = S^{2k-1}/Z_2$. The matrices transforming as a vector, i.e. the X_i, are projected out as well as those which transforming according to any $\mathrm{SO}(2k)$ Young diagram with $\vec{r} = (r_1, 0\ldots)$ and r_1 odd. The advantage of working with RP^{2k-1} is that in the large-n limit, the algebra becomes associative as in the case of the algebra of the fuzzy four-sphere and for similar reasons. The correlation between spatial $\mathrm{SO}(2k)$ transformation properties and a Z_2 subgroup of a U(2) is reminiscent of structures encountered in the study of zero-branes on compact orbifolds in [34, 35]. These analogies may be useful in finding physical brane realizations of fuzzy odd spheres.

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Fuzzy sphere S^D $D = 2k$	n-scaling of Radius $R \sim n$	n-scaling of geometrical dof n^D	n-scaling of total dof $N^2 \sim n^T$ $= n^{k^2+k}$	n-scaling of excess dof n^{T-D}	n-scaling for rank of gauge group $= n^{T-D2}$
S^2	n	n^2	n^2	1	1
S^4	n	n^4	n^6	n^2	n
S^6	n	n^6	n^{12}	n^6	n^3
S^8	n	n^8	n^{20}	n^{12}	n^6

Table 3: Saclings of degrees of freedom for fuzzy even spheres.

Fuzzy sphere S^D $D = 2k-1$	n-scaling of Radius $R \sim n$	n-scaling of geometrical dof n^D	n-scaling of total dof $N^2 \sim n^T$ $= n^{k^2+k-2}$	n-scaling of excess dof n^{T-D}
S^3	n	n^3	n^4	n
S^5	n	n^5	n^{10}	n^5
S^7	n	n^7	n^{18}	n^{11}
S^9	n	n^9	n^{28}	n^{19}

Table 4: Scalings of degrees of freedom for fuzzy odd spheres.

9.3 Combinatoric comparison of even and odd spheres

In the even fuzzy sphere cases we can deduce the rank of the non-abelian group by looking at the geometry. The radius always scales as n. The number of degrees of freedom (dof) required to describe the sphere scales as n^{2k} for S^{2k}. The total number of dof scales as N^2, where N is the size of the matrix, i.e. the dimension of $\mathcal{R}_n$. It is also equal to n^D where D is the dimension of the higher dimensional coset. When we formulate the physics as a non-abelian one, the rank is $n^{D-2k/2}$. This is made concrete in table 3.

It is instructive to compare this with fuzzy odd spheres (table 4)

The first five columns for even and odd spheres are the same, and the odd sphere analog of the last column is conspicuously absent, since we have not been able to decipher the nature of the gauge theory living on the odd sphere, although we did give an outline of a field theory on the higher dimensional coset in section 7. Since the number of extra degrees of freedom scales like an odd power of n, assuming one could make sense of the theory on the odd sphere as some sort of generalized non-abelian gauge theory, the rank would be $n^{p/2}$ for p odd. Curiously, a completely different line of argument, also motivated by an attempt to understand 5-branes, leads one to consider a generalization of non-abelian gauge theory where the number of degrees of freedom scales like k^3 rather than k^2 where k is an integer number of branes. This famous exponent appears in trying to account for the entropy of five-branes [36]. The exponent n^5 should be compared to the odd power of k.[1] It would be interesting to find a scenario where the appearance of odd powers of a brane number as the counting of degrees of freedom of a field theory on a fuzzy odd sphere can be related to the entropy of five-branes.

[1] I thank Savdeep Sethi for discussions along these lines.

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9.4 Further comments and open questions

It is useful to compare the properties of the higher dimensional fuzzy spheres with the fuzzy two-sphere. In the case of the fuzzy two-sphere, we understand not only the finite algebra of functions and how it approxmates, in an SO(3) covariant manner the functions on a classical sphere, but there have also been studies of finite versions of the differential calculus as well as gauge theories on the sphere [37]–[43]. In the case of the even-dimensional fuzzy spheres discussed in [13], it was observed that the full matrix algebras actually describe a higher dimensional geometry $\mathrm{SO}(2k+1)/\,\mathrm{U}(k)$ and a projection is needed resulting in a non-associative algebra for the truncated spectrum of spherical harmonics on the sphere S^{2k}. The methods of the above references should lead to differential calculi for the matrix versions of $\mathrm{SO}(2k+1)/\,\mathrm{U}(k)$. Extending the projection to the differential calculi should be a way to develop the differential calculi for the spheres. In the large-n limit the non-associative sphere becomes an ordinary sphere, and in the case of the four-sphere, there are physical contexts where the dual to the matrix construction is known and as a result aspects of the matrix geometry can be matched with the higher dimensional D-brane gauge theory construction [5, 13]. In the case of the fuzzy odd spheres studied here, the methods developed for the matrix geometry of the fuzzy two-sphere can be expected to be generalizable to the fuzzy versions of the geometries $\mathrm{SO}(2k)/\mathrm{U}(k-1)\times\mathrm{U}(1)$. Again one has to study an appropriate projection to get the finite $\mathrm{SO}(2k)$-covariant geometry of the odd sphere S^{2k-1}. From the discussion in section 9.2 it is apparent that the story is likely to be more subtle than in the even sphere case. We note that other approaches to fuzzy three-spheres have been studied in [44].

The results of [3, 5, 12, 13] show that a lot of interesting physics of spherical 4-branes constructed from 0-branes can be inferred directly from the matrices X_i satisfying the defining equation of an even sphere. It is also clear that the higher dimensional fuzzy spheres give a neat generalization of the physics of fuzzy four-spheres [13, 17]. In this paper, we hope to have convinced the reader that by considering a natural generalization of the fuzzy even sphere construction to the odd sphere case as in [11] we can find many features similar to the fuzzy even sphere case, e.g. the relevance of field theories on higher dimensional cosets. Admittedly these field theories are somewhat more complicated and even the sketch of a reduction to a theory on the sphere itself has not been accomplished (unlike the even sphere case). However the fact that the higher dimensional cosets have a rather rich geometry related to symmetric spaces, as explained in section 9.1 suggests that these fuzzy odd sphere matrix constructions should have a role similar in at least some respects to fuzzy even spheres. One question in the direction of finding this physics is to find a SUSY matrix quantum mechanics model, perhaps allowing the generalized type of SUSY considered in [6], which would admit any of the fuzzy odd spheres as a SUSY preserving solution. It was observed in [6], using simple scaling arguments, that one did not expect a five sphere to appear as a classical solution. It would be interesting to find *any* M-theory background with a matrix model dual where the duality would predict the existence, rather the non existence, of fuzzy five spheres. This may be viewed as a reformulation of the transverse 5-brane problem of BFSS matrix theory.

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A. Notation for Gamma matrices

We recall the explicit construction of the Γ matrices. To fix notation we will give the form of the Gamma matrices Γ_μ with $\mu = 1 \cdots 2k+1$. The Gamma matrices of $\mathrm{SO}(2k+1)$ obey the equations:

$$[\Gamma_\mu, \Gamma_\nu] = \delta_{\mu\nu} \tag{A.1}$$

They can be expressed in terms of a set of fermionic oscillators a_i with i running from 1 to k, and obeying

$$\left\{a_i, a_j^\dagger\right\} = \delta_{ij} \,. \tag{A.2}$$

The expressions for the Gamma matrices are:

$$\begin{aligned} \Gamma_{2i-1} &= \left(a_i + a_i^\dagger\right), \\ \Gamma_{2i} &= i\left(a_i - a_i^\dagger\right) \end{aligned} \tag{A.3}$$

for $i = 1 \dots k$ and $\Gamma_{2k+1} = i^k \Gamma^1 \dots \Gamma^{2k}$. From (A.3) it follows that $\Gamma_{2i-1}\Gamma_{2i} = i[a_i^\dagger, a_i]$.

A 2^k-dimensional representation of the Γ matrix algebra is obtained by defining a state v_0 which is annihilated by the fermionic annihilation operators a_i and acting with the creation operators to generate 2^k different states. Using the abobe we find $\Gamma^5 v_0 = v_0$. States in V_+, the positive chirality representation of $\mathrm{SO}(2k)$, have an even number of $a^\dagger$ acting on v_0. States in V_-, the negative chirality representation of $\mathrm{SO}(2k)$, have an odd number of $a^\dagger$ acting on v_0.

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